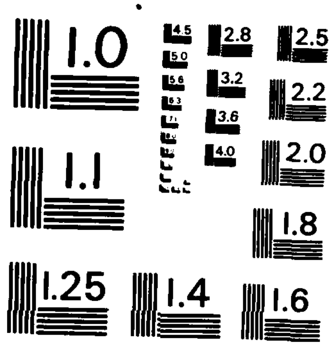
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THE ASYMPTOTIC BEHAVIOUR NEAR THE CREST OF WAVES OF EXTREME FORM

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ABSTRACT

The angle which the free boundary of an extreme wave makes with the horizontal is the solution of a singular, nonlinear integral equation. It has been proved only recently that solutions exist and that (as Stokes suggested in 1880) these solutions represent waves with sharp crests of included angle $2\pi/3$. Amick and Fraenkel have investigated the asymptotic behaviour of the free surface near the crest and obtained an asymptotic expansion for this behaviour, but are unable to say whether the leading term in this expansion has a non-zero coefficient (and so whether it is in fact the leading term or not). The present paper shows that the coefficient is non-zero and determines its sign.

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Key Words: water waves, nonlinear integral equations, asymptotic analysis

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SIGNIFICANCE AND EXPLANATION

This paper concerns waves of permanent form on the free surface of an ideal liquid which is in two-dimensional, irrotational motion under the action of gravity. ~~We~~ It considers only extreme waves, often called "waves of greatest height"; each of these is the end-member of a one-parameter family of waves and is distinguished from other "smaller" members of the family by a sharp crest. Although this corner is physically unrealistic, oceanographers have given such idealised, extreme waves a great deal of attention since Stokes postulated their existence in 1880.

The ~~present~~ paper is a contribution to the strict mathematical theory of extreme waves, which has emerged only since 1978. An asymptotic series is known that describes the flow near the crest, but it has never been proved whether the leading term in this expansion has a non-zero coefficient or not (and so whether it is in fact the leading term or not). The ~~present paper~~ author shows that the coefficient is non-zero and determines its sign. The result should play a useful part in numerical computation of extreme waves.

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THE ASYMPTOTIC BEHAVIOUR NEAR THE CREST OF WAVES OF EXTREME FORM

J. B. McLeod

1. Introduction

This note is in the nature of a postscript to the paper [1], "Behaviour Near the Crest of Waves of Extreme Form," by Amick and Fraenkel. In it they discuss gravity waves, of permanent and extreme form, on the free surface of an ideal liquid, the flow being two-dimensional, irrotational and in a vertical plane. By a wave of extreme form is meant the "largest" member of a one-parameter family of such waves, and it is characterised by a sharp crest of included angle $2\pi/3$. The existence of such waves was conjectured by Stokes in 1880 and recently proved by Amick, Fraenkel and Tolend [2], where a fuller account of the problem is given. (See also [3] for background information)

Amick and Fraenkel are concerned with the asymptotic behaviour of the free surface of the wave near the crest. As is described in [1], the notation of which we adopt, it is a matter of discussing the behaviour as $\zeta \downarrow 0$ of a solution

$$\psi(\zeta) = \frac{1}{3\pi}\int_0^\infty \frac{\omega(\eta)\sin\psi(\eta)}{\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau} \log\left|\frac{\zeta+\eta}{\zeta-\eta}\right| d\eta \quad , \qquad (1.1)$$

where $0 < \zeta < \infty$ and

$$\omega(\eta) = (1+\eta^2)^{-1/2}(1+b\eta^2)^{-1/2} \quad ,$$

the constant b satisfying $0 \le b \le 1$. (The value $b = 0$ corresponds to solitary waves, the values $0 < b < 1$ to periodic waves of finite depth, and the value $b = 1$ to periodic waves of infinite depth.) By a solution of (1.1) is meant a function ψ satisfying (1.1) pointwise and such that $0 < \psi < \frac{1}{2}\pi$, ψ is continuous on $(0,\infty)$, $\psi(\zeta) = O(\zeta^{-1})$ as $\zeta \to \infty$ and $\psi(\zeta) \to \frac{1}{6}\pi$ as $\zeta \downarrow 0$. The existence of such solutions is known (see [1] for references).

It is a standard integral that

$$\frac{1}{3\pi}\int_0^\infty \frac{1}{\eta}\log\left|\frac{\zeta+\eta}{\zeta-\eta}\right| d\eta = \frac{1}{6}\pi \quad , \qquad (1.2)$$

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and so (1.1) and (1.2) together yield

$$\psi(\zeta) - \frac{1}{6}\pi = \frac{1}{3\pi}\int_0^\infty \left(\frac{\omega(\eta)\sin\psi(\eta)}{\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau} - \frac{1}{\eta} \right) \log\left| \frac{\zeta+\eta}{\zeta-\eta} \right| d\eta \quad . \tag{1.3}$$

The basic result of [1] is that

$$\psi(\zeta) - \frac{1}{6}\pi = O(\zeta^{\beta_1}) \quad \text{as} \quad \zeta \downarrow 0 \quad , \tag{1.4}$$

where β_1 is the smallest positive root of the equation

$$\sqrt{3}\,(1+\beta) = \tan\frac{1}{2}\pi\beta \quad , \tag{1.5}$$

so that $0 < \beta_1 < 1$, but it remains possible in [1] that in fact $\psi(\zeta) - \frac{1}{6}\pi$ is of smaller order than ζ^{β_1} . It is the object of the present note to show that this is not so, and that

$$\psi(\zeta) - \frac{1}{6}\pi \sim -A\,\zeta^{\beta_1} \quad \text{as} \quad \zeta \downarrow 0 \quad , \tag{1.6}$$

where A is a strictly positive constant. The method, as in [1] is to use the Mellin transform on equation (1.3). The final result is therefore the following:

<u>Theorem.</u> <u>If</u> ψ <u>is a solution of (1.1) in the sense described, then (1.6) holds</u>.

2. Proof of the Theorem

We define the Mellin transform $\hat{f}$ of a function f in the usual way by setting

$$\hat{f}(s) = \int_0^\infty \zeta^{s-1} f(\zeta)d\zeta \quad .$$

Since from [1] we know that (1.4) holds, the transform $\widehat{(\psi - \frac{1}{6}\pi)}\ (s)$ exists and represents an analytic function of s for $-\beta_1 < \mathrm{re}(s) < 0$. For such s , we may multiply (1.3) by ζ^{s-1} and integrate to obtain

$$\widehat{(\psi - \frac{1}{6}\pi)}\ (s) = \frac{1}{3\pi}\int_0^\infty \left(\frac{\omega(\eta)\sin\psi(\eta)}{\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau} - \frac{1}{\eta} \right) \left(\int_0^\infty \zeta^{s-1}\log\left| \frac{\zeta+\eta}{\zeta-\eta} \right| d\zeta \right) d\eta$$

$$= \frac{1}{3\pi}\left(\int_0^\infty x^{s-1}\log\left| \frac{1+x}{1-x} \right| dx \right) \int_0^\infty \eta^s \left(\frac{\omega(\eta)\sin\psi(\eta)}{\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau} - \frac{1}{\eta} \right) d\eta \ , \tag{2.1}$$

by setting $\zeta = \eta x$. The interchange of order of integration is justified since both

integrals in (2.1) are absolutely convergent. (Note that, by the properties of ψ, the η-integrand is

$$O(\eta^{s-1}) \text{ as } \eta \to \infty \text{ and } O(\eta^{s+\beta_1-1}) \text{ as } \eta \to 0 \text{ .)}$$

If we now use the standard result that the first integral in (2.1) is just $\pi \tan \frac{1}{2}\pi s/s$, and integrate the second integral by parts, we obtain, for $-\beta_1 < \mathrm{re}(s) < 0$,

$$\widehat{(\psi - \tfrac{1}{6}\pi)}(s) = -\frac{1}{3}\tan\frac{1}{2}\pi s \int_0^\infty \eta^{s-1}\log\left(\frac{2}{\eta}\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau\right)d\eta \tag{2.2}$$

$$= -\frac{1}{3}\tan\frac{1}{2}\pi s \int_0^\infty \eta^{s-1}\log\left(\frac{2}{\eta}\int_0^\eta \sin\psi(\tau)d\tau\right)dv - F(s) , \tag{2.3}$$

where

$$F(s) = \frac{1}{3}\tan\frac{1}{2}\pi s \int_0^\infty \eta^{s-1}\log\left(\int_0^\eta \omega(\tau)\sin\psi(\tau)d\tau \Big/ \int_0^\eta \sin\psi(\tau)d\tau\right)d\eta . \tag{2.4}$$

Since $\omega(\tau) = 1 + O(\tau^2)$ as $\tau \to 0$, we see that $F(s)$ is an analytic function of s for $-1 < \mathrm{re}(s) < 0$. But also, since $0 < \psi < \frac{1}{2}\pi$ and $0 < \omega < 1$, we see that $F(s)$ is strictly positive for $-1 < s < 0$. If we further write the logarithm in (2.3) in the form

$$\log\left(1 + \frac{2}{\eta}\int_0^\eta (\sin\psi(\tau) - \tfrac{1}{2})d\tau\right) ,$$

and note that

$$\log(1+t) < t \quad \text{for} \quad t > -1$$

and that

$$\sin\psi - \frac{1}{2} < \frac{1}{2}\sqrt{3}(\psi - \frac{1}{6}\pi) \quad \text{for} \quad 0 < \psi < \frac{1}{2}\pi ,$$

we see that (2.3) becomes

$$\widehat{(\psi - \tfrac{1}{6}\pi)}(s) = -\frac{1}{\sqrt{3}}\tan\frac{1}{2}\pi s \int_0^\infty \eta^{s-2}\left(\int_0^\eta (\psi(\tau) - \tfrac{1}{6}\pi)d\tau\right)d\eta - F_1(s) , \tag{2.5}$$

where $F_1(s)$ is a function analytic for $\max(-1, -2\beta_1) < \mathrm{re}(s) < 0$ and strictly positive for $\max(-1, -2\beta_1) < s < 0$. (In fact, β_1 can be explicitly evaluated from (1.5), and $2\beta_1 > 1$.)

If finally we perform an integration by parts on the integral in (2.5), we obtain, for $-\beta_1 < re(s) < 0$,

$$\left(1 - \frac{1}{\sqrt{3}}\ \frac{\tan \frac{1}{2}\pi s}{s-1}\right) \widehat{\left(\psi - \frac{1}{6}\pi\right)}(s) = -F_1(s) \quad ,$$

or

$$\widehat{\left(\psi - \frac{1}{6}\pi\right)}(s) = -\frac{\sqrt{3}\,(1-s)\,F_1(s)}{\sqrt{3}\,(1-s) + \tan \frac{1}{2}\pi s} \quad ,$$

and we use the inversion theorem on this. Thus

$$\psi(\zeta) - \frac{1}{6}\pi = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{-\sqrt{3}\,(1-s)\,F_1(s)}{\sqrt{3}\,(1-s) + \tan\frac{1}{2}\pi s}\,\zeta^{-s}ds \quad ,$$

where c is any number satisfying $-\beta_1 < c < 0$.

Now move the path of integration from $re(s) = c$ to $re(s) = -\ell$, where $\beta_1 < \ell < 1$. Since the integrand is analytic for $-1 < re(s) < 0$ except for a simple pole at $s = -\beta_1$, where the residue is

$$-\frac{\sqrt{3}\,(1+\beta_1)\,F_1(\beta_1)}{-\sqrt{3} + \frac{1}{2}\pi \sec^2 \frac{1}{2}\pi\beta_1}\,\zeta^{\beta_1} = -A\zeta^{\beta_1} \quad ,$$

say, where $A > 0$, we have

$$\left(\psi - \frac{1}{6}\pi\right)(s) = -A\zeta^{\beta_1} + \lim_{T\to\infty} \frac{1}{2\pi i}\int_{-\ell-iT}^{-\ell+iT} \frac{-\sqrt{3}\,(1-s)\,F_1(s)}{\sqrt{3}\,(1-s) + \tan\frac{1}{2}\pi s}\,\zeta^{-s}ds \ . \tag{2.6}$$

The proof of the theorem is thus complete provided we can show

(i) that the contribution from the horizontal portions of the transformed contour, on $im(s) = \pm T$, are negligible as $T \to \infty$,

and

(ii) that the remaining integral on the right of (2.6) is $O(\zeta^{\ell})$ as $\zeta \to 0$.

To prove (i), consider $F(s)$ as given by (2.4). Since the logarithm is $O(\eta^2)$ as $\eta \to 0$ and $O(\log\eta)$ as $\eta \to \infty$, we see that $|F(s)| \to 0$ as $im(s) \to \pm\infty$, by the Riemann-Lebesgue lemma in its Mellin form, provided that $-1 < re(s) < 0$. A similar remark can be made about $F_1(s)$, and this is suficient to prove (i).

To prove (ii), we can integrate (2.4) by parts to obtain

$$F(s) = -\frac{1}{3s}\tan\frac{1}{2}\pi s\int_0^\infty \eta^s\left(\frac{w(\eta)\sin\psi(\eta)}{\int_0^\eta w(\tau)\sin\psi(\tau)dt} - \frac{\sin\psi(\eta)}{\int_0^\eta \sin\psi(\tau)dt}\right)d\eta \quad .$$

With $re(s) = -\ell$, consider $F(s)$ as a function of $im(s)$. Since the integrand is $0(\eta^{s-1})$ as $\eta \to \infty$ and $0(\eta^{s+1})$ as $\eta \downarrow 0$, Parseval's theorem in its Mellin form tells us that the integral is $L^2(-\infty,\infty)$ as a function of $im(s)$. In view of the factor s^{-1} in front of the integral, we see that $F \in L(-\infty,\infty)$. Similar remarks can be made about F_1 , and it is then clear that the integral in (2.6) is $0(\zeta^\ell)$.

This completes the proof of the theorem.

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